Umklapp collisions and center of mass oscillation of a trapped Fermi gas

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Abstract

Starting from the the Boltzmann equation, we study the center of mass oscillation of a harmonically trapped normal Fermi gas in the presence of a one-dimensional periodic potential. We show that for values of the the Fermi energy above the first Bloch band the center of mass motion is strongly damped in the collisional regime due to umklapp processes. This should be contrasted with the behaviour of a superfluid where one instead expects the occurrence of persistent Josephson-like oscillations.

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It is well known that the center of mass of an atomic cloud confined in a harmonic potential oscillates with the trap frequency irrespective of temperature, inter-particle interactions and quantum statistics (Khon's theorem). This general result is no longer true if the system is also confined by a periodic potential generating barriers separating different wells. While a Bose-Einstein condensate still exhibits a collective behaviour since atoms can tunnel in a coherent way through the barriers [1], a non interacting Fermi gas with Fermi energy lying above the first Bloch band is not able to oscillate around the minimum of the harmonic well [2]. The sudden shift of the harmonic trap indeed causes an asymmetry in the occupation of the left and right orbits which, due to their open nature, persists in time. This causes the system to remain trapped at one side of the harmonic field. The inclusion of interactions changes this scenario favouring the relaxation towards equilibrium [3].

An important question is to understand what is the behaviour of a normal Fermi gas in the collisional regime where the system is in conditions of local equilibrium and the dynamic behaviour is described by the equations of hydrodynamics. In particular the main question addressed in the present letter is whether in the collisional regime the system is able to exhibit center of mass oscillations. The problem is relevant in view of the search of signatures of superfluidity in ultra cold Fermi gases.

We consider a two-component atomic gas trapped by an external potential given by the sum of a harmonic trap of magnetic origin and of a stationary optical potential modulated along the z-axis. The resulting potential is given by

$$V_{ext} = \frac{1}{2}m \left(\omega_\perp^2 r_\perp^2 + \omega_z^2 z^2\right) + sE_R \sin^2 q_B z, \tag{1}$$

where $\omega_\perp$, ω_z are the frequencies of the harmonic trap, $E_R = \hbar^2 q_B^2/2m$ is the recoil energy, q_B being the Bragg momentum and s is a dimensionless parameter providing the intensity of the laser beam. The optical potential has periodicity $d = \pi/q_B$ along the z-axis. In the semiclassical approximation, the energy of the resulting Bloch states is given by $\epsilon(\mathbf{r}, \mathbf{p}) = \epsilon(p_z) + p_\perp^2/2m + V_{ho}(\mathbf{r})$, where $V_{\text{ho}} = m(\omega_\perp^2 r_\perp^2 + \omega_z^2 z^2)/2$ and $\epsilon(p_z)$ is the dispersion relation of the lowest band obtained by solving the 1D Schrodinger equation with the Hamiltonian $H = p_z^2/2m + sE_R \sin^2(q_B z)$. Since $\epsilon(p_z)$ is a periodic function with period $2\hbar q_B$, the quasi-momentum p_z is restricted to the first Brillouin zone defined by $-\hbar q_B \leq p_z \leq \hbar q_B$. We restrict our discussion to low temperatures $T \ll T_F$ and neglect higher bands. We consider situations where the system is in the collisional (hydrodynamic) regime and we calculate the

relaxation rate of the center of mass oscillation due to umklapp collisions. In order to achieve the hydrodynamic condition at such low temperatures, where collisions are suppressed by the Pauli principle, one should increase the value of the scattering length working close to a Feshbach resonance and/or work with very shallow traps along the z-th direction.

We begin our analysis by defining the center of mass coordinates and momenta as $Z = \int zf d\mathbf{p} d\mathbf{r}/h^3$, $P_z = \int p_z f d\mathbf{p} d\mathbf{r}/h^3$ where $f = f_\uparrow + f_\downarrow$ and $f_\uparrow, f_\downarrow$ are the distributions functions of the two spin species which are assumed to be equal ($f_\uparrow = f_\downarrow = f/2$).

By suitable integrations of the Boltzmann equation, one finds the following *exact* equations for the center of mass oscillation:

$$\frac{\partial}{\partial t} Z - \int \frac{\partial \epsilon}{\partial p_z} f \frac{d\mathbf{p} d\mathbf{r}}{h^3} = 0 \,, \tag{2}$$

$$\frac{\partial}{\partial t} P_z + m\omega_z^2 Z = \int p_z C \frac{d\mathbf{p} d\mathbf{r}}{h^3} \,, \tag{3}$$

where C is the collisional integral describing $s-$wave scattering between Bloch states. Eq.(2) directly follows from the equation of continuity, $j_z = \partial \epsilon/\partial p_z$ being the current density along z. In the absence of the periodic potential, the integral in Eq.(3) is zero as momentum is rigorously conserved during collisions. Since in this case the dispersion law reduces to the free value $p_z^2/2m$, one has $\int \partial \epsilon/\partial p_z f d\mathbf{p} d\mathbf{r}/h^3 = P_z/m$ and one recovers the general result $\omega = \omega_z$ for the frequency of the oscillation.

The effect of the optical lattice on the above equations is twofold. First it changes the dispersion relation from the free value $p_z^2/2m$ to $\epsilon(p_z)$. Second, the traslational symmetry is broken and momentum is no longer a good quantum number [4]. This has dramatic consequences for the collisional integral as momentum conservation is replaced by the weaker constraint $p_{1z} + p_{2z} - p_{3z} - p_{4z} = 2\hbar n q_B$, where n is an integer and p_{1z}, p_{2z} and p_{3z}, p_{4z} are, respectively, the initial and final quasi-momenta of the two colliding particles. For two-body interactions, one finds that only umklapp processes with $n = \pm 1$ are allowed.

In an umklapp collision, the system exchanges momentum $\pm 2\hbar q_B$ with the optical lattice. This means that P_z varies in time not only because of the oscillator force $F = -m\omega_z^2 z$ but also as a result of umklapp processes, which act to damp the oscillation. We therefore expect that the right hand side of Eq.(3) is non-zero for out-of-equilibrium distributions.

Let us write the collisional integral as $C(\mathbf{p}, \mathbf{r}) = C^+(\mathbf{p}, \mathbf{r}) - C^-(\mathbf{p}, \mathbf{r})$, where $C^+(\mathbf{p}, \mathbf{r})$ and $C^-(\mathbf{p}, \mathbf{r})$ describes collisions in which one of the particles has, respectively, final and initial momentum $\mathbf{p}$. Introducing the compact notation $d\mathbf{p} \equiv d\mathbf{p}_\perp \Theta(|\hbar q_B| - p_z) dp_z$, where $\Theta(x)$ is

the step function, the C^+ term can be conveniently expressed via the Fermi golden rule in the general form $C^+(\mathbf{p}_1, \mathbf{r}) = \int d\mathbf{p}_2 d\mathbf{p}_3 d\mathbf{p}_4 D$, with

$$
\begin{aligned}
D(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{r}) = \frac{2}{h^6} \frac{2\pi}{\hbar} \\
\sum_{n=0,\pm 1} |U_{fi}^{(n)}|^2 \delta(p_{1z} + p_{2z} - p_{3z} - p_{4z} - 2nq_B) \\
\delta(\mathbf{p}_{\perp 1} + \mathbf{p}_{\perp 2} - \mathbf{p}_{\perp 3} - \mathbf{p}_{\perp 4}) \delta(\epsilon(1) + \epsilon(2) - \epsilon(3) - \epsilon(4)) \\
f_\uparrow(3) f_\downarrow(4) (1 - f_\uparrow(1)) (1 - f_\downarrow(2)),
\end{aligned}
\tag{4}
$$

where $U_{fi}^{(n)}$ is the n−dependent matrix element of the two-body interaction and $\epsilon(j) = \epsilon(\mathbf{r}, \mathbf{p}_j)$. The factor 2 in Eq.(4) comes from the trace over spin indices and f_σ are normalized to $\int f_\sigma d\mathbf{p} d\mathbf{r}/h^3 = N/2$. The C^- term is simply obtained from C^+ by interchanging initial and final states. At thermal equilibrium one has $f_\sigma = f_0 = [e^{\beta(E-\mu)} + 1]^{-1}$ with $\beta = 1/k_B T$ and $E = \epsilon(\mathbf{r}, \mathbf{p})$. In this case $C^+ = C^-$ and hence $C = 0$. The total number of collisions per unit time is given by

$$
\Gamma = \int C^+ \frac{d\mathbf{p}_1 d\mathbf{r}}{h^3} = \Gamma_{nor} + \Gamma_{uk},
\tag{5}
$$

where we have written explicitly the contributions Γ_{nor} and Γ_{uk} coming from normal ($n = 0$) and umklapp ($n = \pm 1$) collisions [see Eq.(4)].

While in the general case Eqs. (2) and (3) are not sufficient to calculate the frequency of the mode, in the hydrodynamic regime one can make the ansatz

$$
f_\sigma(\mathbf{r}, \mathbf{p}, t) = f_0(\epsilon(\mathbf{r}, \mathbf{p}) + u(t) p_z),
\tag{6}
$$

where $u(t)$ is a time dependent function. This corresponds to a rigid displacement of the density current j_z: $j_z(p_z) \longrightarrow j_z(p_z) + u$ and permits to write Eqs (2) and (3) in a closed form.

The ansatz (6) is valid in the limit of strong collisions $\omega\tau \ll 1$, where ω is the frequency of the collective oscillation and τ is a typical collisional time. In the classical regime one has $\tau^{-1} \sim \Gamma/N$. At low temperatures $T \ll T_F$, only the particles near the Fermi surface can be scattered, because of Fermi statistics, and one finds that the hydrodynamic condition takes the form

$$
\frac{1}{\tau} \sim \frac{E_F}{k_B T} \frac{\Gamma}{N} \gg \omega.
\tag{7}
$$

Notice that at low T the rate Γ behaves like $(T/T_F)^3$ [5] so that $1/\tau \sim (T/T_F)^2$ coincides, apart from a numerical factor of the order of unity, with the quasiparticle lifetime calculated at the Fermi surface [6].

Since both the integrated current $\int j_z dz$ and the collisional integral are zero at equilibrium, we expand the ansatz (6) to first order in u and plug the result into Eqs.(2) and (3). Analogously, for the center-of-mass momentum one finds $P_z = \int p_z f d\mathbf{p} d\mathbf{r}/h^3 = u\tilde{m}N$, where

$$\tilde{m} = -\frac{2}{N} \int p_z^2 \frac{\partial f_0}{\partial \epsilon(p_z)} \frac{d\mathbf{p} d\mathbf{r}}{h^3} \tag{8}$$

plays the role of an effective mass for the center of mass oscillation [8]. This permits us to cast Eqs.(2) and (3) in the following closed form, holding in the limit of small amplitude oscillations:

$$\frac{\partial}{\partial t} Z = \frac{P_z}{\tilde{m}}, \tag{9}$$

$$\frac{\partial}{\partial t} P_z + m\omega_z^2 Z = -\frac{P_z}{\tau_{uk}}, \tag{10}$$

where

$$\frac{1}{\tau_{uk}} = -\frac{1}{k_B T} \frac{\int p_{1z}(p_{1z} + p_{2z} - p_{3z} - p_{4z}) D d\mathbf{p}_1 d\mathbf{r}}{\int p_z^2 \partial f_0/\partial \epsilon(p_z) d\mathbf{p}_1 d\mathbf{r}} \tag{11}$$

defines the relevant relaxation time of the oscillation due to umklapp collisions. To derive Eq.(11) we have used the identity $\sum_{j=1}^4 p_j \partial D/\partial \epsilon(p_{jz}) = -\beta(p_{1z} + p_{2z} - p_{3z} - p_{4z})D$, holding at equilibrium also for Fermi statistics. Notice that the contribution from the normal ($n = 0$) collisions, which conserve momentum, identically vanishes. Equations (9) and (10) have the form of a damped harmonic oscillator. Looking for solutions of the form $e^{-i\omega t}$ the dispersion law is given by

$$\omega = -\frac{i}{2\tau_{uk}} \pm \sqrt{\frac{m}{\tilde{m}}\omega_z^2 - \frac{1}{(2\tau_{uk})^2}} \tag{12}$$

showing that the oscillations become overdamped if $\omega_z \tau_{uk} < \frac{1}{2}(\frac{\tilde{m}}{m})^{1/2}$.

By recalling that the integrand D in Eq.(4) is symmetric under interchange $1 \rightleftharpoons 2$ and antysimmetric under interchanges $1 \rightleftharpoons 3$ or $1 \rightleftharpoons 4$, the factor p_{1z} in the numerator can be substituted by the combination $(p_{1z} + p_{2z} - p_{3z} - p_{4z})/4$ which is equal to $\pm\hbar q_B/2$ for ($n = \pm 1$) umklapp processes. Taking into account Eq.(8), Eq.(11) can then be written as

$$\frac{1}{\tau_{uk}} = 4\frac{m}{\tilde{m}}\frac{E_R}{k_B T}\frac{\Gamma_{uk}}{N}, \tag{13}$$

where Γ_{uk} is defined by Eqs.(4) and (5).

Our next goal is to show that, for sufficiently tight optical lattices and gas densities corresponding to $T_F/2\delta > 1$, the condition (7) automatically implies the overdamping of

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the center of mass oscillation. In the following we consider relatively large laser intensities and work in tight-binding approximation. Under this assumption, the energy dispersion is given by $\epsilon(p_z) = \delta\big(1 - \cos[\frac{p_z d}{\hbar}]\big)$ where the bandwidth 2δ is proportional to the tunneling rate between consecutive wells. The Fermi energy is related to the total number of particles by $N = 2\int \Theta(T_F - \epsilon(\mathbf{p}, \mathbf{r}))d\mathbf{p}d\mathbf{r}/h^3$, yielding

$$N = \frac{16}{15\pi^2}\left(\frac{E_R}{\delta}\right)^{1/2}\frac{\delta^3}{\hbar^3\omega_\perp^2\omega_z}\int_{-\pi}^{\pi} h(\tilde{p})^{5/2}\Theta(h(\tilde{p}))d\tilde{p}, \tag{14}$$

where $\tilde{p} = p_z d/\hbar$ and $h(\tilde{p}) = T_F/\delta - 1 + \cos\tilde{p}$. Eq.(14) permits to calculate T_F as a function of the free Fermi energy $T_F^0 = (3N)^{1/3}\hbar(\omega_\perp^2\omega_z)^{1/3}$, evaluated in the absence of the optical lattice, the recoil energy E_R and the bandwidth 2δ. Neglecting higher bands, the Fermi energy T_F is always smaller than the free value T_F^0 reflecting the fact that $\epsilon(p_z) \leq p_z^2/2m$ in the first Brillouin zone.

In order to calculate the relevant interaction matrix elements U_{fi}, we first neglect harmonic trapping. The scattering states are $e^{i\mathbf{P}_\perp \cdot \mathbf{r}_\perp}\psi_k(z)$, where $\psi_k(z)$ is the 1D Bloch wavefunction with quasi-momentum k. We make the ansatz $\psi_k(z) \sim \sum_l e^{ipkd/\hbar}f(z - ld)$, where l labels the wells, f is localized at the origin and normalized to $\int_{d/2}^{d/2}|f(z)|^2 = 1$. In the following we will consider a delta function potential $U(r) = g\delta(r)$, the coupling constant g being related to the scattering length a by $g = 4\pi\hbar^2 a/m$. In the tight-binding limit, the matrix elements are given by

$$U_{fi}^{(\pm 1)} = U_{fi}^{(0)} = gd\int_{-d/2}^{d/2} f^4(z)dz = g\alpha. \tag{15}$$

Conversely, in the absence of the optical lattice the eigenstates of H are simply plane waves and the matrix elements are given by $U_{fi}^{(\pm 1)} = 0$ and $U_{fi}^{(0)} = g$. The factor α of Eq.(15) is larger than one and increases by increasing the laser intensity s, the wavefunction $f(z)$ becoming more and more peaked at the origin. For very large values of s, one can use the asymptotic result $\alpha = \sqrt{\pi/2}s^{1/4}$. This formula actually overestimates the correct value of α for smaller s. As an example, for $s = 5$ it gives $\alpha = 1.9$ while a more accurate calculation gives $\alpha = 1.6$ [7]. From the above discussion, we conclude that the periodic potential enhances interaction effects in two different ways. First it allows for additional (umklapp) collisions to take place. Second, the optical confinement compresses the gas and this results in an increase of the coupling constant from g to αg.

In the presence of harmonic trapping, we can still use result (15) provided local density approximation is applicable. This requires that the trapping frequencies should be small

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compared to the Fermi energy and the bandwith. The collisional rates Γ_{nor} and Γ_{uk} appearing in Eqs.(4) and (5) have been integrated using standard Montecarlo techniques. The ratio Γ_{uk}/Γ_{nor} is plotted in Fig.1 as a function of the temperature for different values of the parameter $T_F/2\delta$. For $T_F \ll 2\delta$, umklapp collisions are negligible at low temperatures. In fact, in this case, the typical initial quasi-momenta p_{1z}, p_{2z} are small compared to $\hbar q_B$ and, due to the energy conservation, this is also true for the final quasimomenta, meaning that processes with $n \neq 0$ are unlike. When the Fermi energy is larger than the bandwith, umklapp processes become instead competitive with the normal ones even at low temperatures and we see that the ratio Γ_{uk}/Γ_{nor} saturates to a constant value. In a uniform system this constant can be analytically evaluated and is equal to $1/2$. In the trapped case one finds a smaller value because the effective Fermi energy $T_F(r) = T_F - V_{ho}(r)$ is $r-$dependent and therefore umklapp collisions are quenched at the edge of the cloud where $T_F(r) < 2\delta$.

We are now ready to compare the overdamping condition with the hydrodynamic condition (7). To this purpose, we have evaluated the effective mass $\tilde{m}$ for the center of mass oscillation introduced in Eq.(8). At $T = 0$ and for a fixed laser intensity s, the ratio $\tilde{m}/m$ is a function of the parameter T_F/E_R. This function is plotted in Fig.(2) for different values of s. The figure shows that, for $T_F/E_R \gtrsim 1$, the ratio $\tilde{m}/m$ does not depends on the laser intensity and is of the order of unity.

In the following we will consider typical configurations with $T_F \sim E_R$. By comparing Eq.(7) with Eq.(13), we conclude that, due to umklapp processes, the center of mass oscillation of a trapped gas confined by a tight optical potential with $T_F > 2\delta$ will be overdamped in the collisional regime $\omega_z \tau \ll 1$ since in this case $\Gamma_{uk} \sim \Gamma_{nor}$, $\tilde{m} \sim m$ and therefore $\tau_{uk} \sim \tau \ll \omega_z^{-1}$.

The relaxation rate of the dipole oscillation can be conveniently written in the form:

$$\frac{1}{\tau_{uk}} = \alpha^2 \frac{a^2}{d^2} \frac{\delta}{\hbar} F\left(\frac{T}{T_F}, \frac{T_F}{2\delta}\right) \tag{16}$$

where the temperature dependence of the function F is plotted in Fig.3 for different values of the paramter $T_F/2\delta$. The possibility for the system to be in the overdamping regime depends on the actual values of the parameters in Eq.(16). As a concrete example, let us consider a two-component gas of $N = 10^5$ potassium (^{40}K) atoms with trap frequencies $\omega_\perp = 2\pi \cdot 275\text{Hz}$ and $\omega_z = 2\pi \cdot 24\text{Hz}$, corresponding to $T_F^0 = 390nK$. For the optical lattice we assume $s = 5$ and periodicity $d = 400\text{nm}$, corresponding to $E_R = 9.2\delta = 374nK$.

For the Fermi energy one then finds $T_F = 0.85T_F^0$ and from the value of $T_F/2\delta$ one finds $\tilde{m}/m = 1.8$. We see from Fig.3 that the overdamping condition $\omega_z\tau_{uk} \ll 1$ is satisfied even at low temperatures $T/T_F \sim 0.05 - 0.1$, provided the scattering length is moderately large, say $|a|/d \geq 0.1 - 0.2$. This can be accomplished experimentally working close to a Feshbach resonance.

For lower temperatures or smaller scattering lengths, the system enters the collisionless regime. Since in this regime the center-of-mass oscillation is self-trapped under the same $T_F > 2\delta$ condition [2], we conclude that the system can never exhibit undamped center-of-mass oscillation in the normal phase. In the superfluid phase, on the contrary, one expects the occurrence of persistent (undamped) Josephson-like oscillations [9]. In particular, in the weak-coupling (BCS) limit, where the distribution function f does not deviate significantly from the ideal gas value, the frequency of the dipole oscillation is expected to be $\omega_z\sqrt{m/\tilde{m}}$ with $\tilde{m}$ given by Eq.(8) and Fig.2.

Let us finally discuss the conditions of applicability of our approach. These concern the stability of the Bloch states with respect to the interaction. First, the scattering length must be small compared to the interwell distance, i.e. $a \ll d$, otherwise we cannot model the interaction with a δ-function potential. Second, the broadening of the Bloch wavefunction due to collisions should be smaller than the bandwith: $\hbar/\tau \ll \delta$. This condition is needed in order to apply the semiclassical Boltzmann picture and can be reasonably well satisfied for the temperatures of interest.

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[8] This mass should not be confused with the effective mass $m^* = 2E_R/\pi^2\delta$ calculated at the bottom of the band. At $T = 0$, one finds $\tilde{m} = m^*$ if $T_F \ll 2\delta$.

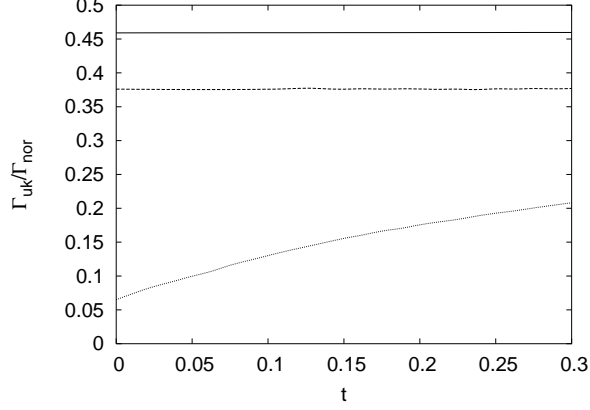[9] M. Wouters, J. Tempere and J.T. Devreese, cond-matt/0312154

FIG. 1: Ratio of umklapp over normal collisional rates [see Eq.(5)] as a function of $t = T/T_F$ for the values $T_F/2\delta = 5$ (solid line), 2 (dashed line), 1 (dotted line).
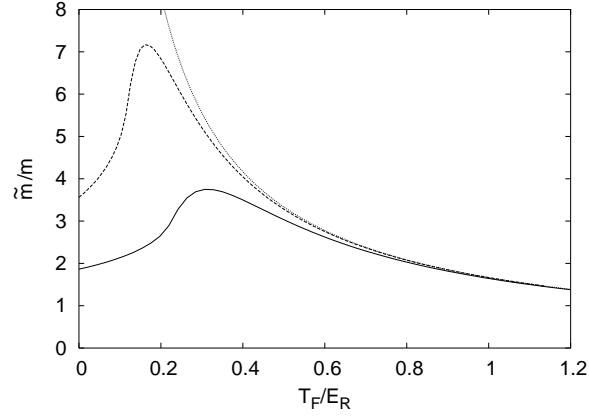

FIG. 2: Effective mass of the dipole oscillation [see Eq.(8)] plotted at $T = 0$ as a function of the parameter T_F/E_R for laser intensities $s = 5$ (solid line) and $s = 8$ (dashed line). The asymptotic curve $5E_R/3T_F$ is also shown (dotted line).
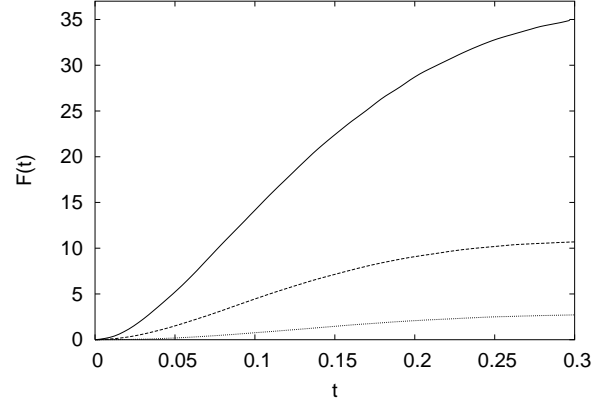
FIG. 3: Dipole relaxation function F [see Eq.(16)] as a function of the reduced temperature $t = T/T_F$ for the values $T_F/2\delta = 5$ (solid line), 2 (dashed line), 1 (dotted line).